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Maximizing Exponential Utility under the Control of New Business

YANG Zhu[1], HUO Zhen-yu[2]

(1- College of Science, Hebei University of Engineering, Handan 056038; 2- School of Information and Electronic Engineering, Hebei University of Engineering, Handan 056038)

Abstract: The insurance company is allowed to invest in new business, and the aim is to maximize the exponential utility of its terminal wealth. The old business and new business are both modelled by compound poisson processes. By using the corresponding Hamilton-Jacobi-Bellman (HJB) equation, the closed-form expressions to the optimal value function and optimal control are derived. These results provide good guidance in practice.

Keywords: compound Poisson process; new business; exponential utility; Hamilton-Jacobi-Bellman equation

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1 Introduction

Recently a number of results have been obtained on stochastic control problems in the insurance literature, using methods based on the HJB equation. These developments stem from Browne[1] and Asumussen[2]. Later there are mainly Asmussen Hϕjaard and Taksar[3], Hipp and Plum[4], Hipp and Taksar[5], Hϕjaard and Taksar[6-8], Schmidli[9,10], Bai and Guo[11].

In this paper, we consider the problem of optimal choice about a new business to maximize the exponential function. It is firstly put forward in Hipp and Taksar[5], where two businesses are both modelled as compound Poisson processes and the objective is to minimize the probability of ruin.

Following the framework of Hipp and Taksar[5], we consider the following surplus process for insurance business (old business)

$$\mathrm{d}R_s^1 = c_1\mathrm{d}s - \mathrm{d}\sum_{i=1}^{N_s^1} X_i, \quad s \in [0,T],$$

where T is a fixed time, $x \geq 0$ denotes the initial surplus, and c_1 denotes the insurer's premium income per unit time, which is collected continuously. The claim counting process $\{N_s^1;\ s \geq 0\}$ is an poisson process with the parameter λ_1, where the inter-occurrence times are denoted by T_i^1, and the claim arrival epochs by $\sigma_n^1 = \sum_{i=1}^n T_i^1$. $X_1, X_2, \cdots$, independent of $\{N_s^1;\ s \geq 0\}$ are positive i.i.d. random variables with common distribution function (df) $G_1(x)$, the moment generating function $\hat{I}_X(x) = E\mathrm{e}^{xX}$, where X is a generic random variable which has the same distribution as $X_i\ (i = 1, 2, \cdots)$.

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Biography: Yang Zhu (Born in 1979), Female, Lecturer. Research field: stochastic process.

For possible new business, we consider the second insurance portfolio modelled by a classical risk process R_s^2

$$\mathrm{d}R_s^2 = c_2\mathrm{d}s - \mathrm{d}\sum_{j=1}^{N_s^2} Y_j, \quad s \in [0,T],$$

with claims intensity λ_2 and premium intensity c_2 which are independent of R_s^1. The inter-occurrence times are denoted by T_i^2, and the claim arrival epochs by $\sigma_n^2 = \sum_{i=1}^n T_i^2$. The claims $Y_1, Y_2, \cdots$ have a common distribution function (df) $G_2(x)$ for the $Y's$ which will differ from the distribution of the $X's$. Its moment generating function is defined by $\hat{I}_Y(x) = E\mathrm{e}^{xY}$.

Define the process R_s

$$R_s = x + (c_1 + c_2)s - \sum_{i=1}^{N_s^1} X_i - \sum_{j=1}^{N_s^2} Y_j, \quad s \in [0,T],$$

let N_s denote its claim counting process, obviously $N_s = N_s^1 + N_s^2$. The inter-occurrence times are denoted by T_i, and the claim arrival epochs by $\sigma_n = \sum_{i=1}^n T_i$.

A strategy α is described by a stochastic process $\{b_s^\alpha;\ 0 \le s \le T\}$, where b_s^α is chosen predictable, i.e., it depends on all information available before s. In our problem, at each point of time s a proportion b_s^α between 0 and 1 of a certain insurance portfolio can be written, or the intensity $b_s'^\alpha$ of acquisition or renewal can be chosen, and this changes the dynamics of the risk business. For a given strategy α for new business, the risk process of the insurer has the following dynamics.

In a short time interval from s to $s+h$:

1) A X-claim occurs with $\lambda_1 h + o(h)$;
2) A Y-claim occurs with probability $\lambda_2 \int_0^h b_{s+t}^\alpha \mathrm{d}t + o(h)$;
3) No claim occurs with probability $1 - \lambda_1 h - \lambda_2 \int_0^h b_{s+t}^\alpha \mathrm{d}t + o(h)$;
4) The amount $c_1 h + c_2 \int_0^h b_{s+t}^\alpha \mathrm{d}t + o(h)$;

Similarly see Hipp and Taksar[5].

Suppose now that the insurer is interested in maximizing the utility function from his terminal wealth, say at time T. The utility function is $u(x)$, typically increasing and concave $(u''(x) < 0)$. For a strategy α, we define the utility attained by the insurer from the state x at time t as

$$V_\alpha(x,t) = E\big[u(R_T^\alpha) \mid R_t^\alpha = x\big],$$

where R_s^α denotes the resulting process controlled by the strategy α. Then our objective is to find the optimal value function

$$V(x,t) = \sup_{\alpha\in\Pi} E\big[u(R_T^\alpha) \mid R_t^\alpha = x\big] \tag{1}$$

and the optimal control α^* such that

$$V(x,t) = V_{\alpha^*}(x,t) = \sup_{\alpha\in\Pi} V_\alpha(x,t),$$

where Π denotes the set of such strategy that is predictable and satisfies $0 \le b_s^\alpha \le 1$ for $s \in [0,T]$.

Suppose that the insurer has an exponential utility function

$$u(x) = n - \frac{\gamma}{m}\mathrm{e}^{-mx}, \tag{2}$$

where $\gamma > 0$ and $m > 0$.

2 Solution of the control problem

In this section, the dynamic programming approach is used to solve the optimization problem. Let C^1 denotes the space of $\phi(x,t)$ such that ϕ and its partial derivatives ϕ_x, ϕ_t are continuous on $R\times[0,T]$. From standard arguments, we know that if the optimal value function $V \in C^1$, then V satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned}\sup_{0\le b\le 1}\Big\{ &\frac{\partial}{\partial t}V(x,t) + (c_1 + bc_2)\frac{\partial}{\partial x}V(x,t)\\ &+ \lambda_1 E\big[V(x-X,t) - V(x,t)\big] + \lambda_2 bE\big[V(x-Y,t) - V(x,t)\big]\Big\} = 0,\end{aligned} \tag{3}$$

with the terminal value

$$V(x,T) = u(x). \tag{4}$$

The following theorem shows that the classical solution to the HJB equation yields the solution to the optimization problem.

Theorem 2.1 Assume that $W \in C^1$ satisfies (3), (4). Then the value function V given by (1) and W coincide. Furthermore, let $B(x,t)$ be such that

$$\begin{aligned}&\frac{\partial}{\partial t}V(x,t) + \big(c_1 + B(x,t)c_2\big)\frac{\partial}{\partial x}V(x,t)\\ &\quad+ \lambda_1 E\big[V(x-X,t) - V(x,t)\big] + \lambda_2 B(x,t)E\big[V(x-Y,t) - V(x,t)\big] = 0,\end{aligned}$$

for all $(x,w,t) \in R\times[0,T]$. Then the control strategy α^* of the form $b_s^{\alpha^*} = B(R_{s-}^{\alpha^*}, s)$ is optimal. That is $W(x,t) = V(x,t) = V_{\alpha^*}(x,t)$.

Proof The proof is similar to that in Hipp and Taksar[5]. Here we omit it.

In view of Theorem 2.1, in order to solve the problem we need to find a C_p^1 solution to the HJB equation (3) and the corresponding maximizing function $B(x,t)$. To this end, suppose that W is found. Since the right-hand side of (3) is a linear function of b, the supremum is always attained at one of the extreme points of $[0,1]$, namely at the point

$$B(x,t) = \begin{cases} 1, & \text{if} \quad \lambda_2 E\big[W(x-Y,t) - W(x,t)\big] + c_2\frac{\partial}{\partial x}W(x,t) \ge 0,\\ 0, & \text{if} \quad \lambda_2 E\big[W(x-Y,t) - W(x,t)\big] + c_2\frac{\partial}{\partial x}W(x,t) < 0.\end{cases} \tag{5}$$

Define the religion

$$\mathcal{C} = \Big\{(x,t) \in R\times[0,T] : \lambda_2 E\big[W(x-Y,t) - W(x,t)\big] + c_2\frac{\partial}{\partial x}W(x,t) \ge 0\Big\}, \tag{6}$$

which will be specified as the final solution is written. For $(x,t) \in \mathcal{C}$, the HJB equation (3) reduces to

$$W_t + (c_1 + c_2)W_x + \lambda_2 E\big[W(x-Y,t) - W(x,t)\big] + \lambda_1 E\big[W(x-X,t) - W(x,t)\big] = 0. \tag{7}$$

To solve the equation (7) with terminal value (4), we will try to fit a solution of the form

$$W(x,t) = n - \frac{\gamma}{m}\mathrm{e}^{-mx+S(t)}, \tag{8}$$

where $S(\cdot)$ is a suitable function. The boundary condition (4) implies that $S(T) = 0$. Inserting this trivial solution (8) into (7), cancelling like terms and rearranging result in

$$S'(t) = m(c_1 + c_2) - \lambda_2\big(\hat{I}_Y(m) - 1\big) - \lambda_1\big(\hat{I}_X(m) - 1\big).$$

Therefore

$$S(t) = \big[m(c_1 + c_2) - \lambda_2\big(\hat{I}_Y(m) - 1\big) - \lambda_1\big(\hat{I}_X(m) - 1\big)\big](t - T). \tag{9}$$

Substituting $W(x,t)$ given through (8) and (9) into (6) yields

$$\mathcal{C} = \begin{cases} R \times [0,T], & \text{if} \quad \lambda\big(\hat{I}_Y(m) - 1\big) \le c_2 m, \\ O, & \text{otherwise}, \end{cases} \tag{10}$$

where O represents empty set.

Redoing the above calculation with $(x,t) \in \overline{\mathcal{C}}$, where

$$\overline{\mathcal{C}} = \Big\{(x,w,t) \in R \times [0,T] : \lambda_2 E\big[W(x-Y,t) - W(x,t)\big] + c_2\frac{\partial}{\partial x}W(x,t) < 0\Big\}, \tag{11}$$

we obtain

$$W(x,t) = n - \frac{\gamma}{m}\mathrm{e}^{-mx+H(t)}, \tag{12}$$

where

$$H(t) = \big[mc_1 - \lambda_1(\hat{I}_X(m) - 1)\big](t - T), \tag{13}$$

Inserting (12) into (11) results in

$$\overline{\mathcal{C}} = \begin{cases} R \times [0,T], & \text{if} \quad \lambda_2(\hat{I}_Y(m) - 1) > c_2 m, \\ O, & \text{otherwise}. \end{cases} \tag{14}$$

By the above statements, we have

Theorem 2.2 If $\lambda_2(\hat{I}_Y(m) - 1) \le c_2 m$ then

$$W(x,t) = n - \frac{\gamma}{m}\mathrm{e}^{-mx+S(t)},$$

is the solution to the HJB equation (3) with the terminal value (4), where $S(t)$ is given by (9). In this case $B(x,t) = 1$.

If $\lambda_2(\hat{I}_Y(m) - 1) > c_2 m$, then

$$W(x,t) = n - \frac{\gamma}{m}\mathrm{e}^{-mx+H(t)},$$

is the solution to the HJB equation (3) with the terminal value (4), where $H(t)$ is given by (13). In this case $B(x,t) = 0$.

Proof One can directly verify that $W(x,t)$ solves the HJB equation (3).

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新业务控制下的最大化指数效用

杨 珠[1]， 霍振宇[2]

(1- 河北工程大学理学院，邯郸 056038; 2- 河北工程大学信息与电气工程学院，邯郸 056038)

摘 要: 假设一个保险公司为了最大化终端财富的效用而进行对一个新业务的投资。原来的业务和新的业务都利用复合 Poisson 过程来描述。通过利用 Hamilton-Jacobi-Bellman (HJB) 方程，得到了最优值函数和最优策略的解析解。所得结果很明确使得它对实践有很好的指导作用。

关键词: 复合 Poisson 过程；新业务；指数效用；Hamilton-Jacobi-Bellman 方程